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Mixed H_2/H_∞ Optimal Control for an Elastic Aircraft

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A mixed H_2/H_∞ optimal control design and its application to a flight control problem of B-1 aircraft is studied. The mixed H_2/H_∞ optimal control design is one of finding an internally stabilizing controller that minimizes the H_2/H_∞ performance index subject to an inequality constraint on H_∞ norm. The application is a linear model of longitudinal motion of B-1 aircraft. A standard eigenvalue problem that involves linear matrix inequalities is formulated, and efficient interior point algorithms are used to solve this problem numerically. Results show that this mixed H_2/H_∞ optimal design leads the designer into a tradeoff between H_2 and H_∞ objectives. Through this method, a designer can determine the controllers that result in the desired closed-loop noise rejection properties and stability robustness.

Nomenclature

a_p = pilot station acceleration, ft/s^2
a_z = c.g. vertical acceleration, ft/s^2
q = pitch rate, rad/s
α = angle of attack, rad/s
γ = H_∞ performance measure
δ_e = elevator command input, rad
λ = mixed H_2/H_∞ performance measure
μ = H_2 performance measure
ξ = first bending mode, in.
$\dot{\xi}$ = first mode rate, in./s
σ = singular value of return difference

Subscripts

b = quantity of rigid body model
e = quantity of elastic model

I. Introduction

THE objective of optimal control design is minimizing or maximizing some index performance by which various designs can be compared. The traditional H_2 optimization attempts to minimize the energy of the system output when the system is faced with white Gaussian noise inputs. The result is a controller adept at handling noises but potentially weak in robustness characteristics and tracking performance. The H_∞ optimization, on the other hand, attempts to minimize the system output energy to unknown but bounded energy inputs. The H_∞ optimal controller results in a highly robust system but one that can be notably deficient in handling noises. The design method explored in this paper is the mixed H_2/H_∞ optimization that handles white Gaussian noise and bounded energy inputs simultaneously and establishes a link between H_2 and H_∞ optimization. Through this method, the designer can determine the tradeoff between the noise rejection (H_2) and robust stability characteristics (H_∞).

Recently, there has been a great deal of interest in formulating a mixed H_2/H_∞ optimal control problem. Although this problem can be stated and motivated quite easily, solving it has turned out to be difficult. Currently, no analytic solution is available. However, very powerful algorithms and associated theory have recently been developed for convex optimization. As a result of this development, we can now very rapidly solve many convex optimization problems for which no traditional analytic solutions are known.

We focus on the mixed H_2/H_∞ problem as formulated by Khargonekar and Rotea.[1] However, we cast this problem as a standard eigenvalue problem that involves linear matrix inequalities (LMI). Efficient interior point algorithms[2] are used to solve this optimization problem numerically.

The plant considered in this study is a linear model for longitudinal motion of B-1 aircraft. The reason why B-1 aircraft is of interest in this study is that the aircraft has a requirement to provide a specified level of ride quality for the crew, since one of the principal missions of this aircraft involves flying for long periods of time in close proximity to the terrain. Using the formulation and techniques already mentioned, we are to find an internally stabilizing controller that minimizes an upper bound for the H_2 norm from the plunge gust input to the pitch rate and the pilot station acceleration while imposing an inequality constraint on the H_∞ norm of another closed-loop transfer function. This problem can be interpreted and motivated as a problem of optimal H_2 performance at the pilot station subject to a robust stability constraint. Because B-1 design requirements have produced a relatively flexible aircraft, two models are of interest in this study. These include a model based on rigid-body dynamics only and one with rigid-body modes and elastic modes included. Numerical results will be provided for these two cases.

The formulation of the mixed H_2/H_∞ optimal control problem is presented in Sec. II. In Sec. III, we cast this constrained optimal control problem into a standard eigenvalue problem that involves linear matrix inequalities. A link between H_2 and H_∞ optimization is also established. In Sec. IV, two design models of B-1 aircraft are presented. These include one based on rigid-body dynamics only and one with elastic modes included in it. In Sec. V, the optimization results for both models are presented. The tradeoff between the noise rejection and robustness performance are discussed. The effects of elastic flexibility on the optimization design are also shown.

II. Mixed H_2/H_∞ Optimal Design Problem

The basic block diagram used in this paper is given in Fig. 1, in which P is the plant and K is the controller. Only finite dimensional linear time-invariant (LTI) systems and controllers will be considered in this paper. The signal w is an exogenous input, u is the

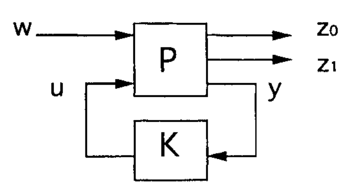

Fig. 1 Block diagram of mixed H_2/H_∞ control.

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control, y is the measurements, and z_0 and z_1 are two sets of outputs of interest. The resulting closed-loop transfer function matrix from w to $z = [z_0 \; z_1]^T$ is denoted by

$$T_{zw} = \begin{pmatrix} T_{z_0 w} \\ T_{z_1 w} \end{pmatrix} \tag{1}$$

The mixed H_2/H_∞ optimal design of interest here is that for a given scalar $\gamma > 0$, we are interested in an internally stabilizing controller K such that an upper bound for $\|T_{z_0 w}(K)\|_2$ is minimized while maintaining $\|T_{z_1 w}(K)\|_\infty < \gamma$. In this paper, we consider a special case in which the states are available for feedback.

Let the plant be given by the following state–space representation:

$$\dot{x} = Ax + B_1 w + B_2 u, \qquad z_0 = C_0 x + D_0 u$$
$$z_1 = C_1 x + D_1 u, \qquad y = x \tag{2}$$

Consider a static controller K such that

$$u = Kx \tag{3}$$

With this controller, the plant (2) becomes

$$\dot{x} = Fx + B_1 w, \qquad z_0 = H_0 x, \qquad z_1 = H_1 x \tag{4}$$

where $F = A + B_2 K$, $H_0 = C_0 + D_0 K$, and $H_1 = C_1 + D_1 K$. The state-feedback gain matrix K should be chosen such that matrix F has all eigenvalues in the open left-half plane.

In this case, the H_2 norm of the closed-loop transfer function matrix $T_{z_0 w}$ is given by

$$\|T_{z_0 w}\|_2^2 = \operatorname{tr}\left(H_0 L_c H_0^T\right) \tag{5}$$

where L_c is the solution of the following Lyapunov equation:

$$F L_c + L_c F^T + B_1 B_1^T = 0 \tag{6}$$

Now, let $\gamma > 0$ be given and assume that $\|T_{z_1 w}\|_\infty < \gamma$. It is known[1,3] that there exists a unique real symmetric matrix Y such that

$$R(Y, \gamma) \triangleq FY + YF^T + Y\frac{H_1^T H_1}{\gamma^2}Y + B_1 B_1^T = 0 \tag{7}$$

and Y satisfies the following inequalities:

$$0 \le L_c \le Y \le \hat{Y} \tag{8}$$

where $\hat{Y}$ represents any real symmetric matrix that satisfies the quadratic matrix inequality $R(\hat{Y}, \gamma) \le 0$. Then,

$$\|T_{z_0 w}\|_2^2 = \operatorname{tr}\left(H_0 L_c H_0^T\right) \le \operatorname{tr}\left(H_0 Y H_0^T\right) \tag{9}$$

So there exists an upper bound for this H_2 norm that we define as the mixed H_2/H_∞ performance index

$$J(T_{zw}, \gamma) \triangleq \left\{\operatorname{tr}\left(H_0 Y H_0^T\right) : Y = Y^T > 0,\, R(Y, \gamma) = 0\right\} \tag{10}$$

Suppose that transfer function matrix $T_{z_0 w}$ is strictly proper and $\|T_{z_1 w}\|_\infty < \gamma$. Then the mixed H_2/H_∞ performance index can be expressed alternatively as

$$J(T_{zw}, \gamma) = \inf_Y \left\{\operatorname{tr}\left(H_0 Y H_0^T\right) : Y = Y^T > 0,\, R(Y, \gamma) < 0\right\} \tag{11}$$

Note that in Eq. (11), $F = A + B_2 K$, $H_0 = C_0 + D_0 K$, and $H_1 = C_1 + D_1 K$. Thus, for a given scalar $\gamma > 0$, the mixed H_2/H_∞ performance index $J(T_{zw}, \gamma)$ in Eq. (11) becomes a function of controller K. The mixed H_2/H_∞ optimal design considered here is to minimize $J(T_{zw}, \gamma)$ by choice of K.

To formulate a convex optimization problem, we replace the search over state-feedback gain matrices K by a search over a larger space of matrices. This is done by introducing the change of variables $K = WY^{-1}$ where Y is a solution to the quadratic matrix inequality $R(Y, \gamma) < 0$ that characterizes the H_∞ norm constraint.

Let the feedback gain $K = WY^{-1}$; the objective function in the performance definition (11) then becomes

$$\begin{aligned}\operatorname{tr}\left(H_0 Y H_0^T\right) &= \operatorname{tr}\left[\left(C_0 + D_0 W Y^{-1}\right) Y \left(C_0 + D_0 W Y^{-1}\right)^T\right] \\ &= \operatorname{tr}\left[(C_0 Y + D_0 W) Y^{-1} (C_0 Y + D_0 W)^T\right] \\ &\triangleq f(W, Y)\end{aligned} \tag{12}$$

The expression of the quadratic constraint $R(Y, \gamma)$ in Eq. (7) becomes

$$\begin{aligned}R(W, Y, \gamma) &= FY + YF^T + Y\frac{H_1^T H_1}{\gamma^2}Y + B_1 B_1^T \\ &= AY + YA^T + B_2 W + W^T B_2^T + B_1 B_1^T \\ &\quad + \frac{1}{\gamma^2}(C_1 Y + D_1 W)^T (C_1 Y + D_1 W)\end{aligned} \tag{13}$$

Finally, the mixed H_2/H_∞ optimal design problem can be formulated as

$$\begin{gathered}\underset{W,Y}{\text{minimize}}\; f(W, Y) \\ R(W, Y, \gamma) < 0, \qquad Y = Y^T > 0\end{gathered} \tag{14}$$

where $f(W, Y)$ and $R(W, Y, \gamma)$ are given by Eqs. (12) and (13), respectively.

This formulation all but follows the one developed by Khargonekar and Rotea.[1] In their formulation, they considered a standard case, $\gamma = 1$. Here we treat γ as a parameter so that we can investigate the effects of γ on this mixed control design.

III. Optimization over LMIs

Consider the mixed H_2/H_∞ optimal control problem (14). The quadratic matrix inequality $R(W, Y, \gamma) < 0$ can be converted into a linear matrix inequality using Schur complements.[2,4]

The basic idea is as follows. The nonlinear matrix inequality,

$$T(x) > 0, \qquad Q(x) - S(x)T(x)^{-1}S(x)^T > 0 \tag{15}$$

where $Q(x) = Q(x)^T$, $T(x) = T(x)^T$, and $S(x)$ depend affinely on x, is equivalent to the following LMI:

$$\begin{pmatrix} Q(x) & S(x) \\ S(x)^T & T(x) \end{pmatrix} > 0 \tag{16}$$

Using this transformation, the quadratic inequality $R(W, Y, \gamma) < 0$ is equivalent to the following LMI:

$$\begin{pmatrix} Q(W, Y) & (C_1 Y + D_1 W)^T \\ C_1 Y + D_1 W & \gamma^2 I \end{pmatrix} > 0 \tag{17}$$

where

$$Q(W, Y) = -AY - YA^T - B_2 W - W^T B_2^T - B_1 B_1^T \tag{18}$$

Introducing an auxiliary symmetric matrix X and an auxiliary scalar λ, then the objective function $f(W, Y)$ in Eq. (12) can be written as follows:

$$\begin{aligned}f(W, Y) = \min \lambda \\ &\lambda - \operatorname{tr}(X) > 0 \\ &X - (C_0 Y + D_0 W) Y^{-1} (C_0 Y + D_0 W)^T > 0 \\ = \min \lambda \\ &\lambda - \operatorname{tr}(X) > 0 \\ &\begin{pmatrix} X & C_0 Y + D_0 W \\ (C_0 Y + D_0 W)^T & Y \end{pmatrix} > 0\end{aligned} \tag{19}$$

Combining Eqs. (17) and (19), expression (14) can then be reduced to a standard eigenvalue problem[2,4] over linear matrix inequalities,

$$\begin{gathered} \underset{W,X,Y}{\text{minimize}}\,\lambda, \qquad \lambda - \operatorname{tr}(X) > 0 \\ \begin{pmatrix} X & C_0Y + D_0W \\ (C_0Y + D_0W)^T & Y \end{pmatrix} > 0 \\ \begin{pmatrix} Q(W,Y) & (C_1Y + D_1W)^T \\ C_1Y + D_1W & \gamma^2 I \end{pmatrix} > 0 \end{gathered} \tag{20}$$

where $Q(W, Y)$ is a linear matrix expression in W and Y that is determined by Eq. (18). Scalar λ takes the place of objective function $f(W, Y)$ as nominal mixed perfomance.

This is a convex optimization problem. We can use the interior point method for the solution. The optimization variables are matrices W, X, and Y. Here, γ appears as a parameter in the problem. Matrices X and Y are symmetric positive definite. Consider the following two extreme cases of problem (20).

Case 1, infinite γ solution: when $\gamma \to \infty$, from the definition of the mixed H_2/H_∞ performance index, we know that

$$\lim_{\gamma\to\infty} J(T_{zw}, \gamma) = \|T_{z_0w}\|_2^2 \tag{21}$$

Thus, the mixed H_2/H_∞ optimization considered here reduces to the standard H_2 optimal control. In this case, the mixed H_2/H_∞ optimization formulation (20) becomes

$$\begin{gathered} \underset{W,X,Y}{\text{minimize}}\,\lambda, \qquad \lambda - \operatorname{tr}(X) > 0 \\ \begin{pmatrix} X & C_0Y + D_0W \\ (C_0Y + D_0W)^T & Y \end{pmatrix} > 0 \\ -AY - YA^T - B_2W - W^TB_2^T - B_1B_1^T > 0 \end{gathered} \tag{22}$$

This is a reduced-order eigenvalue problem over linear matrix inequalities. We can still resort to interior point algorithms for the solutions.

Case 2, minimum γ solution: here we are interested in finding the minimum γ such that the optimization problem (20) is still feasible. This is equivalent to the minimization of the H_∞ norm of the closed-loop transfer fuction T_{z_1w}, resulting in a H_∞ optimal controller. Mathematically, this problem can easily be formulated as

$$\begin{gathered} \underset{W,Y}{\text{minimize}}\,\gamma \\ \begin{pmatrix} Q(W,Y) & (C_1Y + D_1W)^T \\ C_1Y + D_1W & \gamma^2 I \end{pmatrix} > 0 \end{gathered} \tag{23}$$

where linear matrix $Q(W, Y)$ is determined by Eq. (18).

Problem (23) is equivalent to the one formulated by Ghaoui et al.[5] on maximizing a robustness measure for an LTI system with casual nonlinear diagonal perturbations with finite L_2 gain. The solution will be a state-feedback controller that achieves the global maximum of the robustness measure for structured nonlinear perturbations.

Thus, the mixed H_2/H_∞ optimization considered here is between the standard noise rejection and the robustness optimal designs. Through this design technique, it is expected to establish a competition between H_2 and H_∞ goals. To use the interior point algorithms to solve this problem numerically, the constraints of the linear matrix inequalities should be put in the following standard affine form:

$$G(x) = G_0 + \sum_{i=1}^{m} x_i G_i > 0 \tag{24}$$

where $x = [x_1\ x_2\ \cdots\ x_m]^T$ is the variable and $G_i = G_i^T \in R^{N\times N}$, $i = 0, 1, \ldots, m$, are given matrices. This can be done by introducing basis matrices for each matrix variable.

Suppose that the dimensions of matrices A, B_1, B_2, C_0, C_1, D_0, and D_1 in state–space representation (2) are $A \in R^{n\times n}$, $B_1 \in R^{n\times p_1}$, $B_2 \in R^{n\times p_2}$, $C_0 \in R^{q_0\times n}$, $C_1 \in R^{q_1\times n}$, $D_0 \in R^{q_0\times p_2}$, and $D_1 \in R^{q_1\times p_2}$. Then the dimensions of matrix variables X, W, and Y are $X \in R^{q_0\times q_0}$, $W \in R^{p_2\times n}$, and $Y \in R^{n\times n}$.

Let $E_{x_1}, E_{x_2}, \ldots, E_{x_{m_1}}$ be a basis for symmetric $q_0 \times q_0$ matrices $[m_1 = q_0(q_0+1)/2]$, $E_{w_1}, E_{w_2}, \ldots, E_{w_{m_2}}$ a basis for $p_2\times n$ matrices $(m_2 = np_2)$, and $E_{y_1}, E_{y_2}, \ldots, E_{y_{m_3}}$ a basis for symmetric $n \times n$ matrices $[m_3 = n(n+1)/2]$. Then matrix variables X, W, and Y can be expressed as

$$\begin{aligned} X &= x_1E_{x_1} + x_2E_{x_2} + \cdots + x_{m_1}E_{x_{m_1}} \\ W &= w_1E_{w_1} + w_2E_{w_2} + \cdots + w_{m_2}E_{w_{m_2}} \\ Y &= y_1E_{y_1} + y_2E_{y_2} + \cdots + y_{m_3}E_{y_{m_3}} \end{aligned} \tag{25}$$

where $x = [x_1\ x_2\ \cdots\ x_{m_1}]^T$, $w = [w_1\ w_2\ \cdots\ w_{m_2}]^T$, and $y = [y_1\ y_2\ \cdots\ y_{m_3}]^T$ are the variables associated with matrices X, W, and Y, respectively.

Using representation (25), the matrix inequality constraints in problem (20) can be put in the following standard form:

$$G(\lambda, \gamma, x, w, y) = G_0(\lambda, \gamma) + \sum_{i=1}^{m_1} x_iG_i + \sum_{j=1}^{m_2} w_jG_{m_1+j} + \sum_{k=1}^{m_3} y_kG_{m_1+m_2+k} > 0 \tag{26}$$

where $G_i = G_i^T \in R^{N\times N}, i = 0, 1, 2, \ldots, m$ $(m = m_1+m_2+m_3)$ and $N = 1 + q_0 + q_1 + 2n$, are constant matrices.

Then problem (20) can be written as the standard form of an eigenvalue problem over an affine linear matrix inequality so that we can apply the interior point algorithms for the solution,

$$\underset{x,w,y}{\text{minimize}}\,\lambda, \qquad G(\lambda, \gamma, x, w, y) > 0 \tag{27}$$

Both problems (20) and (27) are equivalent forms of the original problem version (14).

In developing interior point algorithms to solve this optimization problem, it is assumed that the problem is feasible and we are given an initial feasible point, i.e., we know λ_0, x_0, w_0, and y_0 for a given γ such that $G(\lambda_0, \gamma, x_0, w_0, y_0) > 0$. The algorithms developed in this paper are based on the method of centers, using a self-concordant barrier function for the linear matrix inequalities. Newton's method, with appropriate step-length selection, is used to efficiently compute the analytic center associated with the barrier function. Details on this method can be found in Ref. 2.

However, finding a feasible point of the linear matrix inequality in Eq. (26) is generally difficult. Currently, no efficient algorithms are available for this LMI feasibility problem.

Here instead of attacking the LMI feasibility problem directly, we consider the feasibility of the original optimization problem (14), that is, finding W_0 and $Y_0 = Y_0^T > 0$ such that $R(W_0, Y_0, \gamma) < 0$. If we know such W_0 and Y_0, then choosing X_0 and λ_0 such that $X_0 - (CY_0 + DW_0)Y^{-1}(CY_0 + DW_0)^T > 0$ and $\lambda_0 - \operatorname{tr}(X_0) > 0$, we obtain a solution to the LMI feasibility problem mentioned.

Consider the feasibility of the following equivalent form of the matrix inequality $R(W, Y, \gamma) < 0$:

$$Q(W, Y) - (1/\gamma^2)(C_1Y + D_1W)^T(C_1Y + D_1W) > 0 \tag{28}$$

where $Q(W, Y)$ is determined by Eq. (18).

Let $\gamma_{\min}$ denote the solution of the H_∞ optimal control problem (23) and assume $\gamma_{\min}$ is finite. Then inequality (28) is feasible for any given $\gamma > \gamma_{\min}$.

Suppose that for a given $\gamma_0 > \gamma_{\min}$, we know an initial feasible point, W_0 and Y_0, of inequality (28). By solving problem (27) using an interior point algorithm, we can obtain a solution to the original optimization problem (14). Let this solution be denoted by W_1 and Y_1. We have the following claims.

1) W_1 and Y_1 satisfy inequality (28) for any $\gamma > \gamma_0$.

2) There exists a $\gamma_1 < \gamma_0$ for given tolerance such that W_1 and Y_1 satisfy the inequality (28) for any $\gamma \in [\gamma_1, \gamma_0]$.

3) Also, $\gamma_{\min}$ is the infimum of this iteration sequence $\{\gamma_k, k = 1, 2, \ldots\}$.

Based on these claims, we develop the following algorithm to solve the mixed H_2/H_∞ optimal control problem formulated.

Algorithm 1: Let W_k and Y_k be the current feasible point of inequality (28) for γ_k.

A1) Pick X_k and λ_k such that

$$X_k - (C_0 Y_k + D_0 W_k) Y_k^{-1} (C_0 Y_k + D_0 W_k)^T > 0 \qquad \lambda_k - \mathrm{tr}(X_k) > 0 \tag{29}$$

A2) Solve problem (27) using interior point algorithms, obtaining W_{k+1} and Y_{k+1} as the solution of the mixed H_2/H_∞ optimal controller for $\gamma = \gamma_k$.

A3) Compute a maximum positive scalar τ_k, such that W_{k+1} and Y_{k+1} are still feasible for $\gamma_{k+1} = \gamma_k - \tau_k$.

A4) Test for convergence for a given tolerance ϵ. If $\tau_k < \epsilon$, the algorithm terminates with γ_{k+1} as the infimum of the iteration sequence. If not, set $W_{k+1} \to W_k$, $Y_{k+1} \to Y_k$, and $\gamma_{k+1} \to \gamma_k$, and go back to step A1.

This algorithm works efficiently if (A, B_2) is stabilizable and (A, C_0) and (A, C_1) are detectable. In this case, τ_k converges to zero monotonically.

Using claim 1 and this algorithm, we can solve the feasibility problem of linear matrix inequality (26) for any $\gamma > \gamma_{\min}$.

IV. Design Example: B-1 Ride Control

We apply this mixed approach to the control of a B-1 aircraft. The design model is the linearized longitudinal equations of motion.[6] The state–space representations are presented as follows:

$$\dot{x} = Ax + B_1 w + B_2 \delta_e, \qquad z_0 = C_0 x + D_0 \delta_e$$
$$z_1 = C_1 x + D_1 \delta_e, \qquad y = x \tag{30}$$

where w is the plunge gust input (feet per second); $z_0 = (q, a_p)^T$, the pitch rate and the pilot station acceleration, and $z_1 = (q, a_z)^T$, the pitch rate and the c.g. vertical acceleration, are two sets of outputs of interest.

By choice of controller, we minimize the H_2 norm of the transfer function $T_{z_0 w}$ to provide ride quality at the pilot station while imposing constraint on the H_∞ norm of the transfer function $T_{z_1 w}$ for stability robustness characteristics.

In present study, we consider two dynamic formulations in Eq. (30) for the mixed optimal design. These include one based on rigid-body dynamics only and one with elastic modes included in it. For both models, we are to determine the controllers that generate the optimal H_2 norm vs H_∞ norm curve, so that we can choose the controller that results in the desired closed-loop noise rejection properties and robustness performance characteristics. For the designs based on the elastic model, we are particularly interested in the effects of the flexibility on this mixed optimal design.

Design Model 1

In this design, the model in Eq. (30) just includes the second-order short-period dynamics of B-1 aircraft. The state variables are angle of attack α and pitch rate q. The specific form of state–space representation (30) is given by

$$\dot{x}_b = \begin{pmatrix} -0.9820 & 0.9595 \\ -10.1149 & -1.6625 \end{pmatrix} x_b + \begin{pmatrix} -0.1523 \\ -11.7993 \end{pmatrix} \delta_e + \begin{pmatrix} 0 \\ -0.0107 \end{pmatrix} w$$
$$z_0 = \begin{pmatrix} 0 & 1 \\ -8.1726 & -4.8607 \end{pmatrix} x_b + \begin{pmatrix} 0 \\ -44.1431 \end{pmatrix} \delta_e$$
$$z_1 = \begin{pmatrix} 0 & 1 \\ 24.0208 & 1.2704 \end{pmatrix} x_b + \begin{pmatrix} 0 \\ 15.5584 \end{pmatrix} \delta_e \tag{31}$$

where $x_b = [\alpha \;\; q]^T$.

Since the norm of matrix B_1 is very small, from the consideration of the numerical computation, we define the nominal plunge gust input $\hat{w}$ as follows:

$$\hat{w} \triangleq 0.0107 w \tag{32}$$

Thus, the plunge gust input term can be written as

$$B_1 w = \begin{pmatrix} 0 \\ -0.0107 \end{pmatrix} w = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \hat{w} = \hat{B}_1 \hat{w} \tag{33}$$

The mixed optimization here is on the transfer function from the nominal plunge gust $\hat{w}$ to the output z.

Design Model 2

In this design, we add the first bending mode and first mode rate into the dynamic equation in state–space model (30). The objective is to investigate the effects of the flexible structure of the aircraft on this mixed optimal design.

This model has four states: angle of attack α, pitch rate q, first bending mode ξ, and first mode rate $\dot{\xi}$. The state–space representation is given by

$$\dot{x}_e = \begin{pmatrix} -0.9896 & 0.9536 & -0.0128 & -0.0001 \\ -10.4128 & -1.8911 & -0.4960 & -0.0150 \\ 0 & 0 & 0 & 1 \\ -228.74 & -175.46 & -380.82 & -4.5800 \end{pmatrix} x_e + \begin{pmatrix} -0.25 \\ -15.66 \\ 0 \\ -2964 \end{pmatrix} \delta_e + \begin{pmatrix} 0 \\ -1.037 \\ 0 \\ -22.69 \end{pmatrix} \hat{w}$$
$$z_0 = \begin{pmatrix} 0 & 1 & 0 & 0.0047 \\ -4.5182 & -2.0576 & 6.0838 & 0.0339 \end{pmatrix} x_e + \begin{pmatrix} 0 \\ 3.2139 \end{pmatrix} \delta_e$$
$$z_1 = \begin{pmatrix} 0 & 1 & 0 & 0.0047 \\ 23.4208 & 0.8102 & -0.9989 & -0.0110 \end{pmatrix} x_e + \begin{pmatrix} 0 \\ 7.7833 \end{pmatrix} \delta_e \tag{34}$$

where $x_e = [\alpha \;\; q \;\; \xi \;\; \dot{\xi}]^T$ and $\hat{w}$ here is nominal plunge gust defined in Eq. (32).

Our numerical procedures for this design start with solving H_2 optimal control problem (22) and terminate with the solution of H_∞ optimal control problem (23) by the following steps.

1) Pick W_0 and $Y_0 = Y_0^T > 0$ such that

$$-AY_0 - Y_0 A^T - B_2 W_0 - W_0^T B_2^T - B_1 B_1^T > 0$$

Then pick X_0 and λ_0 such that inequality (29) is satisfied. These constitute a feasible point of the H_2 optimal control problem (22).

2) Solve the H_2 optimal control problem using interior point algorithms.

3) Using the solution of the H_2 problem as an initial guess, find a feasible point of inequality (28) for a finite γ_0.

4) Initiate Algorithms 1 stated in Sec. III. We obtain a series of the mixed H_2/H_∞ optimal controllers associated with the iteration sequence $\{\gamma_k, k = 1, 2, \ldots\}$. The algorithm terminates with $\gamma_{\min}$, the infimum of the sequence γ_k, as the solution of the H_∞ optimal control.

After obtaining the mixed H_2/H_∞ optimal controllers, instead of studying the mixed optimal performance index λ vs γ, we choose to study the corresponding optimal H_2 norm of the transfer function $T_{z_0 w}$ vs the constraint on the H_∞ norm of the transfer function $T_{z_1 w}$. Using obtained optimal controllers, the optimal H_2 norm of $T_{z_0 w}$ can be computed by

$$\mu \triangleq \|T_{z_0 w}\|_2 = \sqrt{\mathrm{tr}\left[(C_0 + D_0 K) L_c (C_0 + D_0 K)^T\right]} \tag{35}$$

where L_c is the solution of the following Lyapunov equation:

$$(A + B_2 K) L_c + L_c (A + B_2 K)^T + \hat{B}_1 \hat{B}_1^T = 0 \tag{36}$$

where $\hat{B}_1$ is defined in Eq. (33).

V. Optimization Results

The mixed H_2/H_∞ optimal controllers and the associated μ, the optimal H_2 norm of T_{z_0w} for a series of values of γ, the constraint on the H_∞ norm of T_{z_1w}, are obtained for the rigid-body model and the elastic mode model through the discussed algorithms.

The results of two extreme cases of this mixed optimal designs are as follows. 1) H_2 optimal controllers ($\gamma \to \infty$)

$$\left\{ \begin{matrix} \mu_b = 0.6463 \\ K_b = [-0.1871 \quad -0.1074] \end{matrix} \right\} \tag{37}$$

$$\left\{ \begin{matrix} \mu_e = 0.5925 \\ K_e = [-0.4521 \quad -0.2952 \quad 0.4421 \quad 0.0306] \end{matrix} \right\} \tag{38}$$

2) H_∞ optimal controllers (minimum γ)

$$\left\{ \begin{matrix} \mu_b = 8.3046, \quad \gamma_{\min,b} = 1.578 \\ K_b = [1.1638 \quad 0.6888] \end{matrix} \right\} \tag{39}$$

$$\left\{ \begin{matrix} \mu_e = 7.7589, \quad \gamma_{\min,e} = 1.546 \\ K_e = [14.3812 \quad 9.9532 \quad 1.8281 \quad -0.0004] \end{matrix} \right\} \tag{40}$$

The obtained mixed H_2/H_∞ optimal controllers and the associated optimal H_2 norms are between these two extreme cases. To show the performances of these mixed H_2/H_∞ optimal controllers, we plot μ as a function of γ for both models in Fig. 2.

From Fig. 2, one can see the fundamental tradeoff between H_2 and H_∞ objectives in control design. The minimum attainable values of γ are 1.578 for the rigid-body model and 1.546 for the elastic mode model, corresponding to the H_∞ optimal controllers given in Eqs. (39) and (40), respectively. When γ goes up, μ goes down, resulting in controllers adept at handling noises but becoming potentially weak in the H_∞ performances. The minimum attainable values of μ are 0.6463 for the rigid-body model and 0.5925 for elastic mode model, corresponding to the H_2 optimal controllers given by Eqs. (37) and (38), respectively.

From Fig. 2, one can also see that the plot of the optimal H_2 norm of the elastic mode model is below that of the rigid-body model. Thus, using the present models and this mixed optimal design, the elastic aircraft has better performance of noise rejection than rigid-body aircraft for a given level of H_∞ characteristics.

To investigate the stability robustness of these mixed H_2/H_∞ optimal controllers, we compute the minimum singular values of the return difference as a function of frequency by

$$\underline{\sigma}(\omega) = \underline{\sigma}\left[I - K(j\omega I - A)^{-1}B_2\right] \tag{41}$$

Figures 3 and 4 show the plots of $\underline{\sigma}(\omega)$ vs the frequency for different values of γ for the rigid-body model and the elastic mode model, respectively. From Figs. 3 and 4, the following was found.

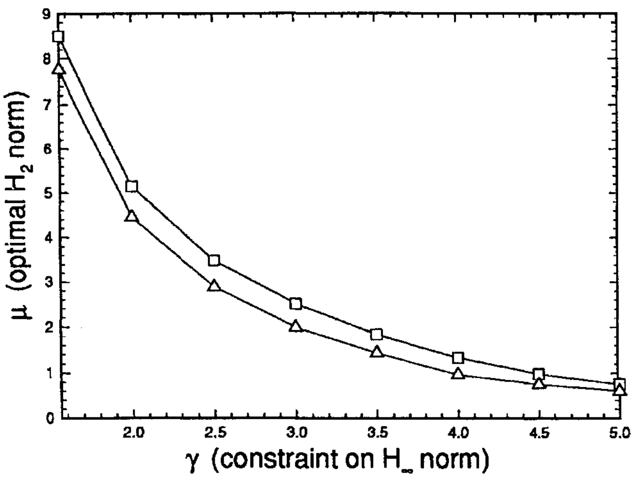

Fig. 2 Optimal H_2 norm vs H_∞ norm: □, μ_b and △, μ_e.

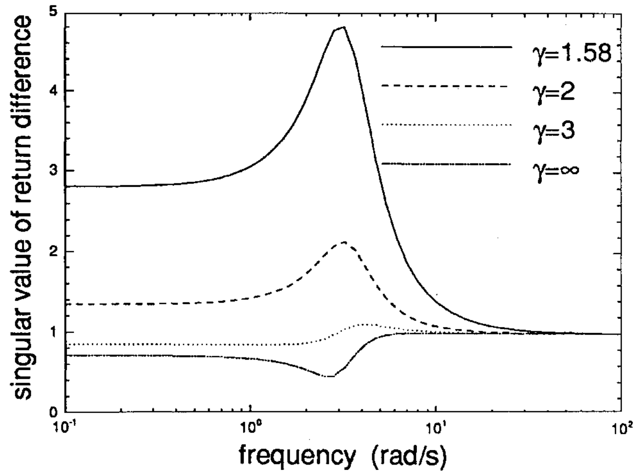

Fig. 3 Singular values of rigid-body aircraft.

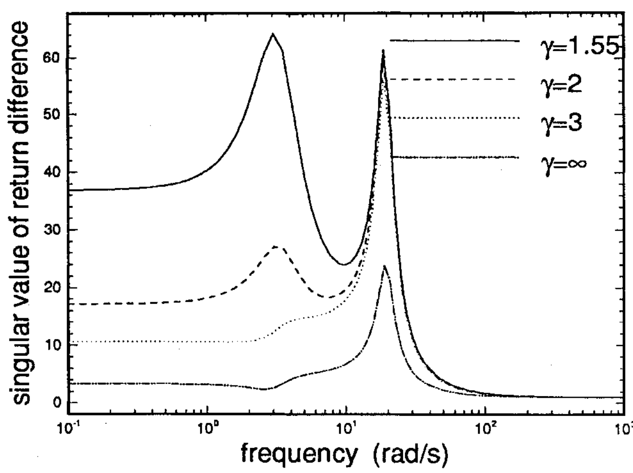

Fig. 4 Singular values of elastic aircraft.

1) When γ goes up, the plot of the singular values of the return difference goes down, resulting in weak stability robustness characteristics. Recalling the results shown in Fig. 2, this is a controller adept at handling noises. When the γ goes down (more strict constraints on the H_∞ norm), the plot of the singular values of the return difference goes up, resulting in strong stability robustness characteristics but the system becomes notably deficient in handling noises. Thus, the designer is faced with a tradeoff between the noise rejection (H_2) and stability robustness performances (H_∞).

Here γ is a performance measure for stability robustness. In Ref. 5, γ^{-1} is used as a robustness performance measure to be maximized for structured nonlinear perturbations.

2) For the H_2 optimal solutions ($\gamma \to \infty$), since matrix C_0 is not orthogonal to matrix D_0, i.e., $C_0^T D_0 \neq 0$, the classic Kalman's inequality[7,8] for standard linear quadratic regulator (LQR) solution no longer exists. This is the general case of H_2 optimal design. In this case, the singular value plot of the return difference will not be guaranteed to be above 1. One can see, from Fig. 3, that the plot of the return difference for the rigid-body model drops below 1. Thus, all of the guaranteed stability properties of the standard LQR are off. But using the mixed H_2/H_∞ optimal design discussed in this paper, we can improve the stability robustness properties of the general H_2 optimal solution at the expense of some H_2 performances.

3) With this mixed H_2/H_∞ optimal controller, the elastic aircraft have bigger singular values of return difference than the rigid-body aircraft, associated with stronger stability robustness characteristics.

More specifically, we can view these characteristics in the time domain by calculating the transient responses of the pilot station acceleration a_p to the unit step input of the plunge gust $\hat{w}$.

Figures 5, 6, and 7 show the step responses of a_p to $\hat{w}$ for H_2, H_∞ optimal designs and a mixed H_2/H_∞ optimal design with $\gamma = 3$, respectively.

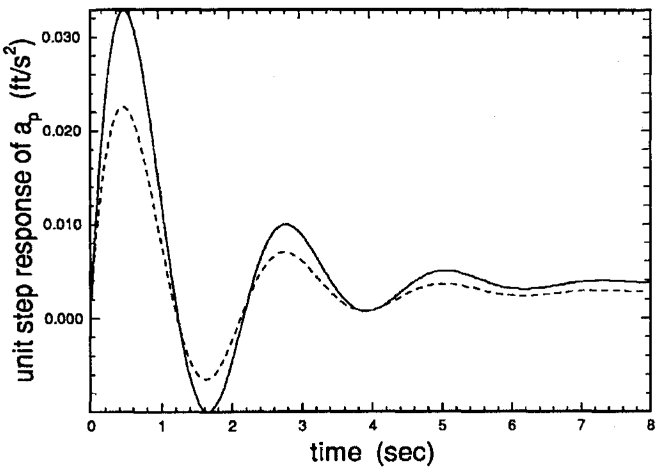

Fig. 5 Step responses of a_p to $\hat{w}$ of H_2 optimal design: ——, rigid body and - - - -, elastic body.

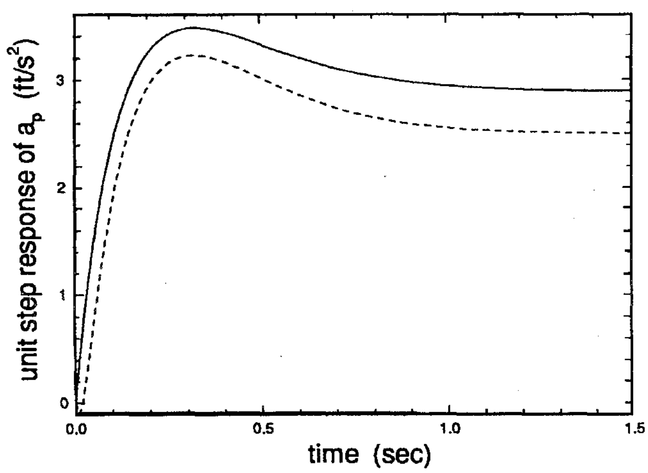

Fig. 6 Step responses of a_p to $\hat{w}$ of H_∞ optimal design: ——, rigid body and - - - -, elastic body.

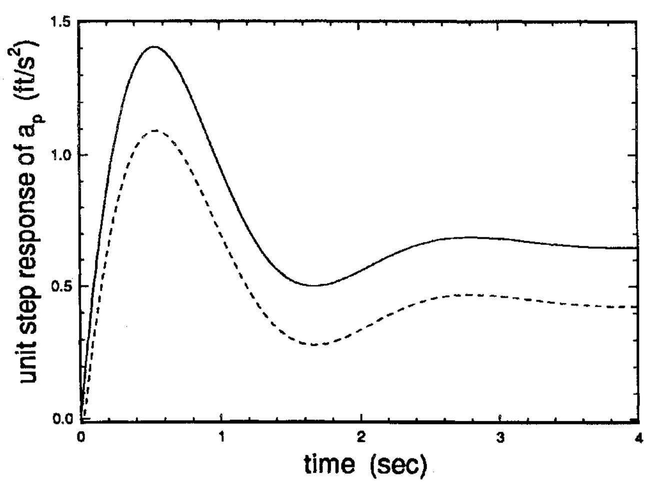

Fig. 7 Step responses of a_p to $\hat{w}$ of mixed H_2/H_∞ optimal design: ——, rigid body and - - - -, elastic body.

From Fig. 5, one can see that the H_2 optimal design can suppress the effects of the plunge gust input significantly because the steady-state outputs of a_p are only 0.0037 (ft/s^2) for the rigid-body model and 0.0027 (ft/s^2) for the elastic mode model. But the resulting system is potentially weak in stability robustness characteristics since the settling time of the transient response is around 6 s.

On the other hand, one can see, from Fig. 6, that the H_∞ optimal design is very strong in stability robustness characteristics because the settling time is only around 1 s, much shorter than that of H_2 optimal design. But the resulting system is notably deficient in handling plunge gust input. The steady-state outputs of a_p are 2.8963 (ft/s^2) for the rigid-body model and 2.5062 (ft/s^2) for the elastic mode model. Actually, the H_∞ optimal design amplifies the plunge gust input on the output.

From Fig. 7, one can see that the mixed H_2/H_∞ optimal design is between these two extreme cases. For $\gamma = 3$, the steady-state outputs of a_p to the unit step input of the plunge gust are 0.6492 (ft/s^2) for the rigid-body model and 0.4250 (ft/s^2) for the elastic mode model. The settling time is around 3 s for both models. Thus, by choice of γ, one can choose the controller that results in the desired closed-loop noise rejection properties and stability robustness characteristics.

From these step responses, one can also see that the step gust does not excite the elastic mode because of this mixed control design.

From all of the preceding, we can see that the elastic aircraft model provides universally better performance than the rigid-body model in this mixed control design. This result, however, may not always be the case. The reasons this result is obtained in the present study are as follows: 1) The models used here are only the linear reduced-order approximate models, and other dynamics and degrading effects are not included; and 2) the elastic model has two more elastic states, and they are assumed to be available for feedback in the design. Therefore, this model has two more degrees of freedom in the design and also more information for feedback. In practice, however, the elastic states are generally not explicitly available for feedback. One is again faced with designing a state estimator. In such a case, the elastic model may not produce as good performance as it does in the state feedback case.

VI. Conclusions

A mixed H_2/H_∞ optimal control design for a flight control problem is explored in this paper. By Schur complements, this mixed optimization problem is formulated as a standard eigenvalue problem that involves linear matrix inequalities. Efficient interior point algorithms are developed to solve this optimization problem numerically. A link between H_2 and H_∞ objectives has been obtained. Using the algorithms developed in this paper, we guarantee a solution to this mixed optimization problem. Through this mixed design, one can optimize the H_2 performance for a specified level of stability robustness characteristics. Or equivalently, we can optimize the robust stability performance for a specified level of the H_2 performance. Thus, the designer can determine the controller that results in the desired noise rejection properties and the robustness characteristic performances.

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